

Answers for the EXAM # 2; MTH 131

1. Using the values given for the function $g(t)$:

- (a) Tell whether the derivative of $g(t)$ appears to be positive or negative. Explain.
- (b) Tell whether the second derivative appears to be positive or negative. Explain.

Answer:

We approximate the derivative, $g'(t)$, at each point t as

$$\frac{g(t+h) - g(t)}{h}.$$

Thus we have

t	100	105	110	115	120
$g(t)$	7.7	6.4	5.0	3.5	1.3
$g'(t)$	$-\frac{1.3}{5}$	$-\frac{1.4}{5}$	$-\frac{1.5}{5}$	$-\frac{2.2}{5}$	
$g''(t)$	$-\frac{0.1}{25}$	$-\frac{0.1}{25}$	$-\frac{0.7}{25}$		

- i. As t increases, $g(t)$ decreases, so $g'(t)$ looks negative.
 - ii. As t increases, $g'(t)$ decreases, so $g''(t)$ also looks negative.
- (c) The national debt of a country is increasing, but the rate at which it increases is slowing down. Sketch a positive graph of the national debt of this country.

Answer:

The graph may look as a logarithm in the first quadrant.

- (d) If $Q(t)$ is the amount of the debt of the country (described in c) at time t , what are the signs of $Q'(t)$ and $Q''(t)$? Explain.

Answer:

$Q'(t)$ is positive and $Q''(t)$ is negative. Compare:

$$f(t) = \ln t, \quad f'(t) = \frac{1}{t} > 0, \quad f''(t) = -\frac{1}{t^2} < 0.$$

2. (a) Find the second derivative of the function $f(t) = 3t^2 + \frac{2}{\sqrt{t}} - \frac{1}{t^2}$.

Answer: We rewrite the function as $f(t) = 3t^2 + 2t^{-1/2} - t^{-2}$. Therefore

$$f'(t) = 6t - t^{-3/2} + 2t^{-3}, \quad f''(t) = 6 + \frac{3}{2}t^{-5/2} - \frac{6}{t^3}.$$

- (b) Find $f'(x)$ if $f(x) = (3 + \sqrt{x}/7)^{14}$.

Answer: We rewrite the function as $f(x) = (3 + \frac{1}{7}x^{1/2})^{14}$. Hence

$$f'(x) = \left(3 + \frac{\sqrt{x}}{7}\right)^{13} \cdot \frac{1}{\sqrt{x}}.$$

3. Find the equation of the line tangent to the graph of $f(x) = \ln(2x - 1)$ at the point where $x = 1$.

Answer:

The line tangent to the graph of $f(x)$ at the point $x = a$ is

$$y - f(a) = f'(a)(x - a).$$

In our case we have $a = 1$, $f(x) = \ln(2x - 1)$, $f'(x) = \frac{2}{2x-1}$ and therefore $f(1) = 0$ and $f'(1) = 2$. Substitution into the formula gives the line tangent to be

$$y = 2(x - 1).$$

4. Differentiate:

(a) $y = 2e^{-3x} \cos(x)$.

Answer:

$$y' = -6e^{-3x} \cos(x) - 2e^{-3x} \sin(x).$$

(b)

$$f(t) = \frac{1 - t^2}{1 + t^2}.$$

Answer:

$$y' = -\frac{4t}{(1 + t^2)^2}.$$

5. Let $y = 3x^5 - 5x^3$.

(a) Find all critical points of y .

Answer:

Critical points are $x = 0$, $x = -1$, and $x = 1$ since

$$y' = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x - 1)(x + 1).$$

(b) Find the intervals where y' is positive. On what intervals is y increasing?

Answer:

The function $y(x)$ increases on the interval where $y' > 0$ which corresponds to $x^2 > 1$. Therefore

$$x > 1 \quad \text{or} \quad x < -1.$$

The function $y(x)$ decreases on the interval

$$-1 < x < 1.$$

6. Suppose the graph below is that of the derivative of a function f . For the interval $-1 \leq x \leq 2$, find the critical points and tell where the function is increasing and where it is decreasing.

Answer:

Critical points are the points where $f'(x)$ is equal to zero. Therefore they are -1 , 0 , 1 , 2 . The function $f(x)$ is decreasing where the graph is above the x -axis (corresponding to where $f'(x)$ is positive) i.e., on the intervals $(-1, 0)$ and $(1, 2)$ and increasing on the interval $(0, 1)$.

7. Find the derivative of the following functions:

(a) $y = e^{-3t^2+2t}$.

Answer: $y' = (-6t + 2)e^{-3t^2+2t}$.

(b) $y = x \ln(2x + 1)$.

Answer: $y' = \ln(2x + 1) + \frac{2x}{2x+1}$.

8. The number of hours, H , of daylight in Madrid is given by the formula

$$H = f(t) = 12 + 2.4 \sin(0.0172t - 1.376),$$

where t is the number of days since the beginning of the year. For example, today, November 8, would correspond to $t = 313$.

(a) Find the rate of change of H with respect to t .

Answer:

$$f'(t) = 2.4 \cos(0.0172t - 1.376) \cdot (0.0172).$$

(b) Find $f(313)$ and $f'(313)$ and explain what information each of these answers gives you.

Answer:

The number of hours of daylight in Madrid on November 8 is

$$f(313) = 12 + 2.4 \sin(0.0172 \cdot 313 - 1.376) \approx 10.17.$$

The rate of change of the number of daylight hours on November 8 is

$$f'(313) = 2.4 \cos(0.0172 \cdot 313 - 1.376) \cdot (0.0172) \approx -0.027.$$

So daylight hours on November 9 will be approximately 0.027 hour (=1.62 min) less than on November 8.

9. Let

$$f(x) = \frac{\sin 3x}{x}.$$

Find $\lim_{x \rightarrow 0} f(x)$. You may use your calculator, but explain how you found the answer.

Answer: Using values of x near 0, we see that

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{x} = 3.$$

You could also examine the graph of the function near 0.

10. Let $f(x) = 2x^2 - x$. Then $f'(x) = 4x - 1$. Show this by using the definition of the derivative.

Answer:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Since $f(x+h) = 2(x+h)^2 - (x+h) = 2x^2 + 4xh + h^2 - x - h$ we have

$$f(x+h) - f(x) = 4xh + h^2 - h.$$

Therefore

$$f'(x) = \lim_{h \rightarrow 0} \frac{4xh + h^2 - h}{h} = \lim_{h \rightarrow 0} (4x + h - 1) = 4x - 1.$$